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CONTROL EFFECTS ON THE ELASTIC BLADE OF A HELICOPTER ROTOR

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CONTROL EFFECTS ON THE ELASTIC BLADES OF A HELICOPTER ROTOR */1073

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The problem of forced vibrations of the blade of a helicopter rotor is investigated, where the perturbation forces are due to the effect of the control (at a variation with time of the collective setting angle, respectively of the amplitude of cyclic variation). The effect of the control during abrupt changes as well as the same effect during gradual oblique changes in command are taken into consideration. It is emphasized that an integration of the equations of motion of the blade is necessary, by means of a Laplace transformation; the conditions required for attenuation or complete disappearance of these vibrations are given.

Notations

OXYZ = Fixed system of axes (XOY-plane coinciding with the control plane)

Oryz = System of rotary axes (Or-axis coinciding with the position of the rigid blade)

R = Radius of the rotor

t_v = Chord of the blade

r = Distance of a given section of the rotor axis

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* Numbers in the margin indicate pagination in the original foreign text.

ζ_n = Generalized coordinate

z = Deviation of the tested blade from the position of the rigid blade

ω = Angular velocity of the rotor

ω_0 = Angular velocity before control

$\Delta\omega$ = Variation in angular velocity

ν_n = Natural frequency of the rotary blade

w = Induced velocity

w_0 = Induced velocity before control

Δw = Variation of the induced velocity

M_g = Generalized mass of the blade

θ = Angle of attack of the blades

θ_0 = Angle of attack before control

$\Delta\theta$ = Variation of the angle of attack

α_i = Induced angle

β = Flapping angle of the blades

C'_z = Slope of the curves referring to the blade profile /1074

$\Phi_n(r)$ = Function of modal form

ρ = Atmospheric density

p = Complex quantity

$h(t)$ = Object function

$$g(p) = p \int_0^\infty e^{-pt} h(t)dt$$

Studies on the vibrations suffered by a blade at a collective angle of attack, on the amplitude of cyclic variation and constant angular velocity, have shown that the perturbation forces can, in the general case, be expressed in the form of a Fourier series.

In the case of a collective angle of attack, the amplitude of cyclic variation and the angular velocity become a function of time, and it might happen,

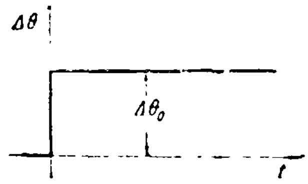

Fig.1

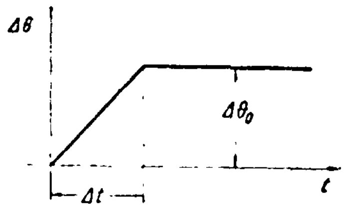

Fig.2

at the instant at which a command is given, that high perturbation forces are generated which exert their effect on the elastic blades, a point which is the

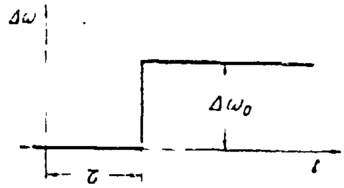

Fig.3

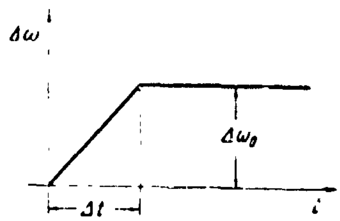

Fig.4

object of our studies.

With respect to the collective angle of attack and the amplitude of cyclic variation, we have considered two of the many types of variations with time as they occur in practice, namely, the variation during sudden application of control (Fig.1) and the variation during gradual oblique application (Fig.2).

Since the helicopter command for the collective setting angle is coupled with the gas pedal for the engine, it can be assumed that, because of these two types of time variation, the angular velocity of the rotor changes slightly during sudden application of control (Fig.3) or completely during gradual application.

In these diagrams, the evolution of the deflection is shown, taking into consideration the case of deflection without asymmetry of the blade velocity (vertical deflection or fixed point). This is known as the "in situ" command.

It is of interest to see here that, under these conditions, the de- /1075
formation of the elastic blade due to the control effect, together with other types of transitory dynamic response of the blade, are attenuated, which is of practical value in the flying of helicopters.

I. EFFECTS DURING SUDDEN APPLICATION OF CONTROL

a. Effect of Control on the Collective Angle of Attack

Let us adopt the hypothesis of uniform induced velocity in the mentioned evolution and, for the time being, let us assume average constant values in the calculation, so that with respect to one blade element the distance r can be generally written as

$$dP = \frac{\rho}{2} t_p (\omega r)^2 C_a' (\theta - \alpha_i)\, dr = \frac{\rho}{2} t_p (\omega r)^2 C_a' \left(\theta - \frac{w}{\omega r}\right) dr. \qquad (1)$$

The effect of control manifests itself here by a rise in angular velocity, and the angle of attack becomes a function of time.

Through the variation in time, the boundaries near the blade influenced by the control give rise to the generation or displacement of vibrations which incite deflections

$$z = \sum_{n=1}^{\infty} \zeta_n \Phi_n \tag{2}$$

which represents a criterion for the position of the rigid blade, which position can be determined from the aerodynamic data of the rotor (Fig.5).

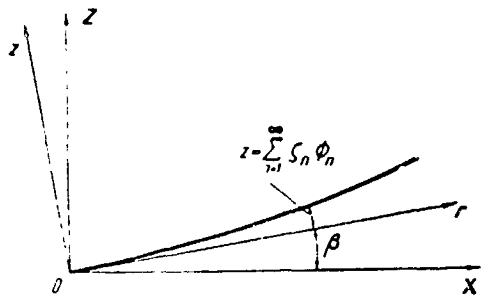

Fig.5

In accordance with the data given in another report (Bibl.2), the equation of motion of the blade has the form

$$\zeta_n + a\,\dot{\zeta}_n + b\,\zeta_n = A\,F_{0n}, \tag{3}$$

in which

$$a = \frac{\pi \rho \omega \int_0^R t_p\, r\, \Phi_n(r)\, dr}{M_n + \frac{\pi\rho}{4}\int_0^R t_p^2\, \Phi_n(r)\, dr},$$

$$b = \frac{M_n \nu_n^2}{M_n + \frac{\pi\rho}{4}\int_0^R t_p^2\, \Phi_n(r)\, dr},$$

$$A = \frac{1}{M_n + \frac{\pi\rho}{4}\int_0^R t_p^2\, \Phi_n(r)\, dr}.$$

Owing to the effects of control it is obvious that then the coefficient a /1076
and b of eq.(3) become functions of time and that, in this sense, we can put

$$\omega = \omega_0 + \Delta\omega$$

and, again, the natural frequency of the rotating blade will be

$$\nu_n^2 = \nu_0^2 + k_1 \omega^2 = \nu_0^2 + k_1 (\omega_0 + \Delta\omega)^2 = (\nu_0^2 + k_1 \omega_0^2) + 2 k_1 \nu_0 \Delta\omega + k_1 \Delta\omega^2,$$

With $\theta = \theta_0 + \Delta\theta$, eq.(3) becomes

$$\ddot{\zeta}_n + [\alpha_0 + \alpha_1(t)]\dot{\zeta}_n + [\varepsilon_0 + \varepsilon_1(t) + \varepsilon_2(t)]\zeta_n = \beta_0 + \beta_1(t) \tag{4}$$

or

$$\ddot{\zeta}_n + \alpha_0 \dot{\zeta}_n + \alpha_{10} \dot{\zeta}_n (t-\tau) + \varepsilon_0 \zeta_n + \varepsilon_{10} \zeta_n (t-\tau) + \\ + \varepsilon_{20} \zeta_n (t-\tau) = \beta_0 + \beta_1 (t-\tau). \tag{5}$$

Let us apply to eq.(5) the Laplace transformation, together with the limit conditions

$$\dot{\zeta}_n(0) = 0,$$
$$\zeta_n(0) = 0,$$

Without diminishing the generality, we will then have

$$\begin{aligned}
&\mathrm{L}\zeta_n = X, \\
&\mathrm{L}\varepsilon_0 \zeta_n = \varepsilon_0 X, \\
&\mathrm{L}\alpha_0 \dot{\zeta}_n = \alpha_0 pX, \\
&\mathrm{L}\ddot{\zeta}_n = p^2 X, \\
&\mathrm{L}\alpha_{10} \dot{\zeta}_n (t-\tau) = \alpha_{10} e^{-p\tau} \mathrm{L}\dot{\zeta}_n = \alpha_{10} e^{-p\tau} pX, \\
&\mathrm{L}\varepsilon_{10} \zeta_n (t-\tau) = \varepsilon_{10} e^{-p\tau} X, \\
&\mathrm{L}\varepsilon_{20} \zeta_n (t-\tau) = \varepsilon_{20} e^{-p\tau} X, \\
&\mathrm{L}\beta_0 = \beta_0, \\
&\mathrm{L}\beta_1 (t-\tau) = \beta_{10} e^{-p\tau}.
\end{aligned}$$

Since p is a complex quantity and since the exponential function can be expanded in a series with infinite convergence domain, we have

$$e^{-p\tau} = 1 - p\tau + \frac{p^2 \tau^2}{2} + \ldots \tag{6}$$

Summing the first two terms of this expansion and making the necessary calculations, eq.(5) becomes

$$X\left[p^2+\frac{\alpha_0+\alpha_{10}-\tau\varepsilon_{10}-\tau\varepsilon_{20}}{1-\alpha_{10}\tau}p+\frac{\varepsilon_0+\varepsilon_{10}+\varepsilon_{20}}{1-\alpha_{10}\tau}\right]=$$

$$=\frac{\beta_0+\beta_{10}}{1-\alpha_{10}\tau}-\frac{\beta_0\tau}{1-\alpha_{10}\tau}p, \tag{7}$$

where /1077

$$\alpha_0=\frac{\pi\rho\,\omega_0\int_0^R t_p\,r\,\Phi_n(r)\,dr}{M_n+\frac{\pi\rho}{4}\int_0^R t_p^2\,\Phi_n(r)\,dr},$$

$$\alpha_{10}=\frac{\pi\rho\,\Delta\omega_0\int_0^R t_p\,r\,\Phi_n(r)\,dr}{M_n+\frac{\pi\rho}{4}\int_0^R t_p^2\,\Phi_n(r)\,dr},$$

$$\varepsilon_0=\frac{M_n(\nu_0^2+k_1\,\omega_0^2)}{M_n+\frac{\pi\rho}{4}\int_0^R t_p^2\,\Phi_n(r)\,dr},$$

$$\varepsilon_{10}=\frac{2\,M_n\,k_1\,\nu_0\,\Delta\omega_0}{M_n+\frac{\pi\rho}{4}\int_0^R t_p^2\,\Phi_n(r)\,dr},$$

$$\varepsilon_{20}=\frac{M_n\,k_1\,\Delta\omega_0^2}{M_n+\frac{\pi\rho}{4}\int_0^R t_p^2\,\Phi_n(r)\,dr},$$

$$\beta_0=\frac{C_a'\frac{\rho}{2}\,\omega_0^2\,\theta_0\int_0^R\Phi_n(r)\,t_p\,r^2\,dr-C_a'\frac{\rho}{2}\,\omega_0\,w\int_0^R\Phi_n(r)\,t_p\,r\,dr}{M_n+\frac{\pi\rho}{4}\int_0^R t_p^2\,\Phi_n(r)\,dr},$$

$$\beta_{10}=\frac{H_0\frac{\rho}{2}C_a'\int_0^R\Phi_n(r)\,t_p\,r^2\,dr-C_a'\frac{\rho}{2}\,\Delta\omega_0\,w\int_0^R\Phi_n(r)\,t_p\,r\,dr}{M_n+\frac{\pi\rho}{4}\int_0^R t_p^2\,\Phi_n(r)\,dr}.$$

$$H_0=2\omega_0\,\theta_0\,\Delta\omega_0+\Delta\omega_0^2\,\theta_0+\omega_0^2\,\Delta\theta_0+2\omega_0\,\Delta\omega_0\,\Delta\theta_0+\Delta\omega_0^2\,\Delta\theta_0.$$

Using the notations,

$$2\,B=\frac{\alpha_0+\alpha_{10}-\tau\varepsilon_{10}-\tau\varepsilon_{20}}{1-\alpha_{10}\tau},$$

$$C^2=\frac{\varepsilon_0+\varepsilon_{10}+\varepsilon_{20}}{1-\alpha_{10}\tau},$$

$$D = \frac{\beta_0 + \beta_{10}}{1 - \alpha_{10}\tau},$$

$$E = \frac{\beta_0 \tau}{1 - \alpha_{10}\tau},$$

we obtain from eq.(7) /1078

$$X = \frac{D}{p^2 + 2Bp + C^2} - \frac{Ep}{p^2 + 2Bp + C^2}. \tag{8}$$

Making an inverse transformation will yield the solution of eq.(5):

$$\zeta_n = \frac{D}{C^2}\left[1 - \frac{C}{\sqrt{C^2 - B^2}} e^{-Bt} \sin\left(\sqrt{C^2 - B^2}\, t + \Phi\right)\right] - \frac{E}{\sqrt{C^2 - B^2}} e^{-Bt} \sin \sqrt{C^2 - B^2}\, t \tag{9}$$

for $C^2 > B^2$, and

$$\zeta_n = \frac{D}{C^2}\left[1 - \frac{C^2}{\Lambda_1 - \Lambda_2}\right]\left(\frac{e^{-\Lambda_2 t}}{\Lambda_2} - \frac{e^{-\Lambda_1 t}}{\Lambda_1}\right) - \frac{E}{\Lambda_1 - \Lambda_2}\left(e^{-\Lambda_2 t} - e^{-\Lambda_1 t}\right) \tag{10}$$

for $C^2 < B^2$.

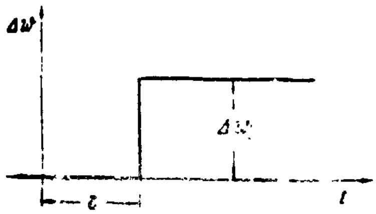

Fig.6

In this expression

$$\Phi = \operatorname{arc\,tg} \frac{\sqrt{C^2 - B^2}}{B}$$

Λ_1 and Λ_2 are again roots of the equation $p^2 + 2Bp + C^2 = 0$.

In the case of combining the limit conditions in the more general form of

$$\zeta_n(0) = \zeta_0,$$
$$\dot{\zeta}_n(0) = 0$$

and, combining the variation of the induced velocity as shown in Fig.6, we will have

$$\mathrm{L}\ \zeta_n = X,$$
$$\mathrm{L}\ \varepsilon_0\, \zeta_n = \varepsilon_0\, X,$$
$$\mathrm{L}\ \alpha_0\, \dot{\zeta}_n = \alpha_0\, p\, X - \alpha_0\, p\, \zeta_0,$$
$$\mathrm{L}\ \ddot{\zeta}_n = p^2 X - p^2 \zeta_0,$$
$$\mathrm{L}\ \alpha_{10}\, \dot{\zeta}_n (t-\tau) = \alpha_{10}\, e^{-p\tau}\ \mathrm{L}\ \dot{\zeta}_n = \alpha_{10}\, e^{-p\tau}\, pX - \alpha_{10}\, e^{-p\tau}\, p\, \zeta_0,$$
$$\mathrm{L}\ \varepsilon_{10}\, \zeta_n\, (t-\tau) = \varepsilon_{10}\, e^{-p\tau}\, X,$$
$$\mathrm{L}\ \varepsilon_{20}\, \zeta_n\, (t-\tau) = \varepsilon_{20}\, e^{-p\tau}\, X,$$
$$\mathrm{L}\ \beta_0 = \beta_0,$$
$$\mathrm{L}\ \beta_1\, (t-\tau) = \beta_{10}\, e^{-p\tau},$$

where

$$\beta_0 = \frac{C_a' \frac{\rho}{2}\, \omega_0^2\, \theta_0 \int_0^R \Phi_n(r)\, t_p\, r^2\, dr - C_a' \frac{\rho}{2}\, \omega_0\, w_0 \int_0^R \Phi_n(r)\, t_p\, r\, dr}{M_n + \frac{\pi\rho}{4} \int_0^R t_p^2\, \Phi_n(r)\, dr},$$

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$$\beta_{10} = \frac{H_0 \frac{\rho}{2}\, C_a' \int_0^R \Phi_n(r)\, t_p\, r^2\, dr - U_0 \frac{\rho}{2}\, C_a' \int_0^R \Phi_n(r)\, t_p\, r\, dr}{M_n + \frac{\pi\rho}{4} \int_0^R t_p^2\, \Phi_n(r)\, dr},$$

$$U_0 = \omega_0\, \Delta w_0 + w_0\, \Delta\omega_0 + \Delta\omega_0\, \Delta w_0.$$

With our conditions, eq.(5) then becomes

$$X\left[p^2 + \frac{\alpha_0 + \alpha_{10} - \varepsilon_{10}\tau - \varepsilon_{20}\tau}{1 - \alpha_{10}\tau}\, p + \frac{\varepsilon_0 + \varepsilon_{10} + \varepsilon_{20}}{1 - \alpha_{10}\tau}\right] = \frac{\beta_0 + \beta_{10}}{1 - \alpha_{10}\tau} + \frac{\alpha_0\, \zeta_0 - \beta_{10}\tau + \alpha_{10}\tau}{1 - \alpha_{10}\tau}\, p + \zeta_0\, p^2, \tag{11}$$

from where, taking account of the notations from the preceding case, we will obtain

$$X = \frac{D}{p^2 + 2Bp + C^2} + \frac{Fp}{p^2 + 2Bp + C^2} + \frac{\zeta_0\, p^2}{p^2 + 2Bp + C^2}, \tag{12}$$

where

$$F = \frac{\alpha_0\, \zeta_0 - \beta_{10}\tau + \alpha_{10}\tau}{1 - \alpha_{10}\tau}.$$

Solution of eq.(9) will thus be completed with the inverse transformation of $\frac{p^2}{p^2 + 2Bp + C^2}$ which will yield the expression

$$-\frac{C}{\sqrt{C^2 - B^2}}\, e^{-Bt} \sin\left(\sqrt{C^2 - B^2}\, t - \Phi\right) \tag{13}$$

if $C^2 > B^2$, and

$$\frac{1}{\Lambda_1 - \Lambda_2}(\Lambda_1 e^{-\Lambda_1 t} - \Lambda_2 e^{-\Lambda_2 t}) \tag{14}$$

if $C^2 < B^2$.

On further examination of the solution for both cases of the limit conditions, it will be found that the quantity ζ_n, independent of time, becomes very small as soon as the quantities B and C become very large.

This reduces to the statement that the condition of vibration caused by the control effect is greatly attenuated as soon as the quantity α_{10} becomes of the order of $\frac{1}{\tau}$.

In this manner, it can be readily demonstrated that a connection exists between the blade parameters and the response time of the engine, such that the initiation of commands does not cause forced vibrations of the blade.

b. Effect of Cyclic Variation on the Angle of Attack /1080

In this case, the angular velocity remains constant ($\omega = \omega_0 = \text{const}$) and only the angle of attack undergoes a cyclic variation so that, taking this into consideration in the usual manner, this angle can be considered as having the form of $\Delta\theta \sin \omega t$.

Let us neglect the transitory phase and, consequently, also the induced angle of flapping motion and, at the same time, take into consideration

$$\theta = \theta_0 + \Delta\theta \sin \omega t, \tag{15}$$

where $\Delta\theta$ follows from the abrupt discontinuity in the slope of the curve in Fig.1, so that the differential equation of motion of the blade will be

$$\ddot{\zeta}_n + a\dot{\zeta}_n + b\zeta_n = \bar{\beta}_0 + \bar{\beta}_1(t). \tag{16}$$

The coefficients a and b in the above equation are expressed here as well as in eq.(3) but only if we assume here that $\omega = \omega_0$ = const.

Thus, the predicted quantities $\bar{\beta}_0$ and $\bar{\beta}_1(t)$ can be expressed as follows:

$$\bar{\beta}_0 = \beta_0 = \frac{C_a' \frac{\rho}{2} \omega_0^2 \int_0^R \Phi_n(r)\, t_p\, r^2 \left(\theta_0 - \frac{w}{\omega_0 r}\right) \mathrm{d}r}{M_n + \frac{\pi\rho}{4} \int_0^R t_p^2\, \Phi_n(r)\, \mathrm{d}r},$$

$$\bar{\beta}_1(t) = \frac{C_a' \frac{\rho}{2} \omega_0^2\, \Delta\theta_0 \int_0^R \Phi_n(r)\, t_p\, r^2\, \mathrm{d}r}{M_n + \frac{\pi\rho}{4} \int_0^R t_p^2\, \Phi_n(r)\, \mathrm{d}r} \sin \omega t = K_0 \sin \omega t.$$

Let us apply the Laplace transformation of eq.(16), together with the limit conditions

$$\zeta_n(0) = \zeta_0,$$
$$\dot{\zeta}_n(0) = 0,$$

which will yield

$$\mathrm{L}\ \ddot{\zeta}_n = p^2 X - p^2 \zeta_0,$$
$$\mathrm{L}\ \dot{\zeta}_n = p X - p\, \zeta_0.$$

In order to preserve the notations of the preceding case, let us put a = = 2B and b = C^2.

Then, eq.(16) becomes

$$p^2 X - p^2 \zeta_0 + 2\, Bp\, X - 2\, Bp\, \zeta_0 + C^2 X = \beta_0 + K_0 \frac{p\, \omega}{p^2 + \omega^2},$$

from which it follows that /1081

$$X = \zeta_0 \frac{p^2}{p^2 + 2Bp + C^2} + 2B\zeta_0 \frac{p}{p^2 + 2Bp + C^2} + \frac{\beta_0}{p^2 + 2Bp + C^2} + K_0\, \omega \frac{p}{(p^2 + \omega^2)(p^2 + 2Bp + C^2)}. \tag{17}$$

Because of the fact that the first three terms have been inversely transformed, it is relatively easy to make use of Tables. Thus, it is possible to control the terms if we write the expression in the following manner:

$$K_0\,\omega \frac{p}{(p^2+\omega^2)\,(p^2+2Bp+C^2)} = \frac{K_0\,\omega \dfrac{p^2}{p^2+\omega^2}\cdot\dfrac{1}{p^2+2Bp+C^2}}{p}.$$

Let us apply the Borel theorem, arranged so that

$$g_1(p) = \mathbf{L}\, h_1(t),$$
$$g_2(p) = \mathbf{L}\, h_2(t),$$

simultaneous with

$$\frac{g_1(p)\,g_2(p)}{p} = \mathbf{L}\int_0^t h_1(T)\,h_2(t-T)\,\mathrm{d}T = \mathbf{L}\int_0^t h_2(T)\,h_1(t-T)\,\mathrm{d}T.$$

We will then have

$$\mathbf{L}^{-1} K_0\,\omega \frac{p^2}{p^2+\omega^2} = K_0\,\omega\cos\omega t,$$

$$\mathbf{L}^{-1}\frac{1}{p^2+2Bp+C^2} = \begin{cases} \dfrac{1}{C^2}\left[1-\dfrac{C}{\sqrt{C^2-B^2}}\,\mathrm{e}^{-Bt}\sin\left(\sqrt{C^2-B^2}\,t+\Phi\right)\right] & \text{if}\quad C^2 > B^2, \\[2ex] \dfrac{1}{C^2}\left[1-\dfrac{C^2}{\Lambda_1-\Lambda_2}\left(\dfrac{\mathrm{e}^{-\Lambda_2 t}}{\Lambda_2}-\dfrac{\mathrm{e}^{-\Lambda_1 t}}{\Lambda_1}\right)\right] & \text{if}\quad C^2 < B^2, \end{cases}$$

where Λ_1, Λ_2 are roots of the equation $p^2 + 2Bp + C^2 = 0$.

Let us consider the first case $C^2 > B^2$ and, for facilitating the calculation, let us introduce the notations

$$k = \sqrt{C^2-B^2},\quad u = \frac{K_0\,\omega}{C^2},\quad v = \frac{K_0\,\omega}{Ck}.$$

Then,

$$\int_0^t h_2(T)\,h_1(t-T)\,\mathrm{d}T =$$
$$= \int_0^t \left[u\cos\omega(t-T) - v\,\mathrm{e}^{-BT}\sin(kT+\Phi)\cos\omega(t-T)\right]\mathrm{d}T.$$

After performing higher integration, we can write /108

$$\mathbf{L}^{-1}\frac{K_0\,\omega\dfrac{p^2}{p^2+\omega^2}\cdot\dfrac{1}{p^2+2Bp+C^2}}{p} =$$

$$= \frac{u}{\omega} \sin \omega t - \frac{v}{2} [\cos \Phi \cos \omega t \, (I_1 + I_2) +$$

$$+ \sin \Phi \cos \omega t \, (J_1 + J_2) + \cos \Phi \sin \omega t \, (J_2 - J_1) + \tag{18}$$

$$+ \sin \Phi \sin \omega t \, (J_1 - J_2)],$$

where

$$I_1 = -\frac{B}{B^2 + \omega_1^2} e^{-Bt} \sin \omega_1 t - \frac{\omega_1}{B^2 + \omega_1^2} e^{-Bt} \cos \omega_1 t + \frac{\omega_1}{B^2 + \omega_1^2},$$

$$I_2 = -\frac{B}{B^2 + \omega_2^2} e^{-Bt} \sin \omega_2 t - \frac{\omega_2}{B^2 + \omega_2^2} e^{-Bt} \cos \omega_2 t + \frac{\omega_2}{B^2 + \omega_2^2},$$

$$J_1 = -\frac{B}{B^2 + \omega_1^2} e^{-Bt} \cos \omega_1 t + \frac{\omega_1}{B^2 + \omega_1^2} e^{-Bt} \sin \omega_1 t + \frac{B}{B^2 + \omega_1^2},$$

$$J_2 = -\frac{B}{B^2 + \omega_2^2} e^{-Bt} \cos \omega_2 t + \frac{\omega_2}{B^2 + \omega_2^2} e^{-Bt} \sin \omega_2 t + \frac{B}{B^2 + \omega_2^2},$$

$$\omega_1 = k + \omega,$$

$$\omega_2 = k - \omega.$$

In that of the above two cases in which again $C^2 < B^2$, and using the notations

$$\gamma_1 = \frac{K_0 \omega}{\Lambda_1 (\Lambda_1 - \Lambda_2)}, \quad \gamma_2 = \frac{K_0 \omega}{\Lambda_2 (\Lambda_1 - \Lambda_2)}$$

we will have

$$\int_0^t h_2 (T) \, h_1 (t - T) \, dT =$$

$$= \int_0^t [u \cos \omega (t - T) + \gamma_1 e^{-\Lambda_1 T} \cos \omega (t - T) - \gamma_2 e^{-\Lambda_2 T} \cos \omega (t - T)] \, dT.$$

After integration, this yields

$$L^{-1} \frac{K_0 \omega \dfrac{p^2}{p^2 + \omega^2} \cdot \dfrac{1}{p^2 + 2Bp + C^2}}{p} =$$

$$= -\left(\frac{\Lambda_1 \gamma_1}{\Lambda_1^2 + \omega^2} e^{-\Lambda_1 t} - \frac{\Lambda_2 \gamma_2}{\Lambda_2^2 + \omega^2} e^{-\Lambda_2 t} \right) + \tag{19}$$

$$+ \left(\frac{\gamma_1 \Lambda_1}{\Lambda_1^2 + \omega^2} - \frac{\gamma_2 \Lambda_2}{\Lambda_2^2 + \omega^2} \right) \cos \omega t + \left(\frac{\gamma_1 \omega}{\Lambda_1^2 + \omega^2} - \frac{\gamma_2 \omega}{\Lambda_2^2 + \omega^2} \right) \sin \omega t.$$

II. EFFECT OF CONTROL IN OBLIQUE GRADUAL STEPS /1083

a. Effect of Control on the Collective Angle of Attack

In this case, starting from a summary analysis of the control effects, we are taking into consideration the variation in the angular velocity $\Delta\omega$, in the angle $\Delta\theta$, and in the induced velocity Δw as shown in Figs.4, 2, and 7.

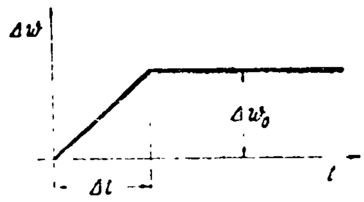

Fig.7

Then, the natural frequency will have an approximately constant value of the following form:

$$\nu_{nm} = \nu_{nc} + k\,\Delta\,\omega_{0m}^2,$$

where $\Delta\omega_{0m}$ is the average value, so that the equation of motion of the blade can be written as follows:

$$\ddot{\zeta}_n + (a_0 + a_1 t)\,\dot{\zeta}_n + b\,\zeta_n = B_0 + B_1 t + B_2 t^2 + B_3 t^3, \tag{20}$$

where

$$a_0 = \frac{\pi\,\rho\,\omega_0 \int_0^R t_r\, r\, \Phi_n(r)\, dr}{M_n + \dfrac{\pi\rho}{4}\int_0^R t_p^2\, \Phi_n(r)\, dr},$$

$$a_1 = \frac{\pi\rho\,\dfrac{\Delta\,\omega_0}{\Delta t}\int_0^R t_p\, r\, \Phi_n(r)\, dr}{M_n + \dfrac{\pi\rho}{4}\int_0^R t_p^2\, \Phi_n(r)\, dr},$$

$$b = \frac{M_n\,\nu_{nm}}{M_n + \dfrac{\pi\rho}{4}\int_0^R t_p^2\, \Phi_n(r)\, dr},$$

$$B_0 = \frac{C_a'\,\dfrac{\rho}{2}\int_0^R \left(\omega_0^2\,\theta_0 - \dfrac{w_0}{r}\,\omega_0\right)\Phi_n(r)\, t_r\, r^2\, dr}{M_n + \dfrac{\pi\rho}{4}\int_0^R t_p^2\, \Phi_n(r)\, dr},$$

$$B_1 = \frac{C_a' \frac{\rho}{2} \int_0^R \left(2\,\omega_0\,\theta_0 \frac{\Delta\,\omega_0}{\Delta t} + \omega_0^2 \frac{\Delta\,\theta_0}{\Delta t} - \omega_0 \frac{\Delta\,w_0}{r\,\Delta\,t} - w_0 \frac{\Delta\,\omega_0}{r\,\Delta\,t)}\right) \Phi_n(r)\,t_p\,r^2\,\mathrm{d}r}{M_n + \frac{\pi\rho}{4} \int_0^R t_p^2\,\Phi_n(r)\,\mathrm{d}r},$$

$$B_2 = \frac{C_a' \frac{\rho}{2} \int_0^R \left(\theta_0 \frac{\Delta\,\omega_0^2}{\Delta\,t^2} + 2\,\omega_0 \frac{\Delta\,\omega_0\,\Delta\theta_0}{\Delta\,t^2} - \frac{\Delta\,\omega_0\,\Delta w_0}{r\,\Delta\,t^2}\right) \Phi_n(r)\,t_p\,r^2\,\mathrm{d}r}{M_n + \frac{\pi\rho}{4} \int_0^R t_p^2\,\Phi_n(r)\,\mathrm{d}r},$$ /1084

$$B_3 = \frac{C_a' \frac{\rho}{2} \frac{\Delta\,\omega_0^2\,\Delta\,\theta_0}{\Delta t^3} \int_0^R \Phi_n(r)\,t_p\,r^2\,\mathrm{d}r}{M_n + \frac{\pi\rho}{4} \int_0^R t_p^2\,\Phi_n(r)\,\mathrm{d}r}.$$

Applying eq.(20) of the Laplace transforms under the limit conditions

$$\zeta_n(0) = 0,$$
$$\dot{\zeta}_n(0) = 0,$$

we will have

$$\mathrm{L}\; b\,\zeta_n = b\,X,$$
$$\mathrm{L}\; a_0\,\dot{\zeta}_n = a_0\,p\,X,$$
$$\mathrm{L}\; a_1\,t\,\ddot{\zeta}_n = -\,a_1\,p\,\frac{\mathrm{d}X}{\mathrm{d}p}\cdot$$
$$\mathrm{L}\; \ddot{\zeta}_n = p^2\,X,$$
$$\mathrm{L}\; B_n\,t^n = B\frac{n!}{p^n},$$

and, after substitution, we will obtain the differential equation

$$\frac{\mathrm{d}X}{\mathrm{d}p} + P\,X = Q, \tag{21}$$

where

$$P = -\left(\alpha_1\,p + \alpha_2 + \alpha_3\,\frac{1}{p}\right),$$

$$Q = -\frac{1}{a_1}\left(\frac{B_0}{p} + \frac{B_1}{p^2} + \frac{2\,B_2}{p^3} + \frac{6\,B_3}{p^4}\right),$$

$$\alpha_1 = \frac{1}{a_1},\quad \alpha_2 = \frac{a_0}{a_1},\quad \alpha_3 = \frac{b}{a_1}\cdot$$

The solution of eq.(21) will then be

$$X = C\,p^{\alpha_3}\,e^{\frac{\alpha_1}{2}p^2}\,e^{\alpha_2 p} +$$
$$+\,p^{\alpha_3}\,e^{\frac{\alpha_1}{2}p^2}\,e^{\alpha_2 p}\int \frac{1}{p^{\alpha_3}\,e^{\frac{\alpha_1}{2}p^2}\,e^{\alpha_2 p}}\left[-\,\alpha_1\left(\frac{B_0}{p} + \frac{B_1}{p^2} + \frac{2\,B_2}{p^3} + \frac{6\,B_3}{p^4}\right)\right]\mathrm{d}p. \tag{22}$$

On expanding this expression in a series of exponential functions and summing a determined number of terms, and considering that α_3 is not an integer, we will obtain terms of the function X being a sum of terms of the form $k_j p^n$ (n differing from 1, 2, 3), so that this inverse transformation represents a sum of terms of the form $k_j \frac{t^{-n}}{\Gamma(1-n)}$. From this result, a first condition is obtained because of the fact that the amplitude of vibration tends toward zero, which necessarily means that the function Γ also tends toward infinite, meaning that, simultaneously, α_3 is a whole number. /1085

Here, it can be demonstrated that the two terms, together with a considerable restriction in the terms of the exponential series, can be calculated by inverse transformation with the aid of the Borel theorem.

b. Effect of Cyclic Variation on the Angle of Attack

Here and in the preceding case, the angular velocity remains constant ($\omega = \omega_0 =$ const) while the amplitude of the cyclic variation depends linearly on the time.

Thus, the angle of attack can be expressed by

$$\theta = \theta_0 + \Delta\theta \sin \omega t,$$

where

$$\Delta\theta = \frac{\Delta\theta_0}{\Delta t} t.$$

The equation of motion of the blade will then be

$$\ddot{\zeta}_n + a\dot{\zeta}_n + b\zeta_n = \beta_0 + q_1(t), \tag{23}$$

where the quantities a, b, and β_0 in the above expression are the same as given in eq.(12) where again $q_1(t)$ is expressed by

$$q_1(t) = -\frac{C_a' \frac{\rho}{2} \omega^2 \frac{\Delta\theta_0}{\Delta t} \int_0^R \Phi_n(r)\, t_p r^2\, dr}{M_n + \frac{\pi\rho}{4} \int_0^R \Phi_n(r)\, t_p^2\, dr} t \sin \omega t =$$

$$= \frac{C_a' \rho \, \omega^3 \frac{\Delta \vartheta_0}{\Delta t} \int_0^R \Phi_n(r) \, t_p \, r^2 \, dr}{M_n + \frac{\pi \rho}{4} \int_0^R \Phi_n(r) \, t_p^2 \, dr} \cdot \frac{t}{2\omega} \sin \omega t = q_0 \frac{t}{2\omega} \sin \omega t.$$

A Laplace transformation, at the initial conditions

$$\zeta_n(0) = \zeta_0,$$
$$\dot{\zeta}_n(0) = 0,$$

and making use of eq.(23) will yield, as also follows from the previous case, the imaginary function

$$X = \zeta_0 \frac{p^2}{p^2 + 2Bp + C^2} + 2B\zeta_0 \frac{p}{p^2 + 2Bp + C^2} + \frac{\beta_0}{p^2 + 2Bp + C^2} + q_0 \frac{p^2}{(p^2 + \omega^2)^2 (p^2 + 2Bp + C^2)}, \qquad (24)$$

where $a = 2B$ and $b = C^2$.

The problem of deriving the inverse transformation can then be arbitrarily expressed by the ultimate terms which can be written in the following form: /1086

$$q_0 \frac{p^2}{(p^2 + \omega^2)^2 (p^2 + 2Bp + C^2)} = \frac{q_0 \frac{p^3}{(p^2 + \omega^2)^2} \cdot \frac{1}{p^2 + 2Bp + C^2}}{p}$$

making use of the Borel theorem.

Because of

$$L^{-1} q_0 \frac{p^3}{(p^2 + \omega^2)^2} = \frac{q_0}{2\omega} (\sin \omega t + \omega t \cos \omega t)$$

we will obtain the inverse transformation of the quantity

$$\frac{1}{p^2 + 2Bp + C^2}$$

in the above-discussed case, so that we will have, at $C^2 > B^2$,

$$L^{-1}\frac{q_0\dfrac{p^3}{(p^2+\omega^2)^2}\cdot\dfrac{1}{p^2+2Bp+C^2}}{p}=\int_0^t\{u\,[\sin\omega\,(t-T)+$$
$$+\,\omega\,(t-T)\cos\omega(t-T)-v\,e^{-Bt}\sin(kT+\Phi)\,[\sin\omega\,(t-T)+ \qquad (25)$$
$$+\,\omega\,(t-T)\cos\omega\,(t-T)]\}\,dT,$$

where

$$k=\sqrt{C^2-B^2},\quad u=\frac{q_0}{2\omega C^2},\quad v=\frac{q_0}{2\omega Ck}.$$

In the case of $C^2 < B^2$, we will have together with

$$\bar{\gamma}_1=\frac{q_0}{2\omega\Lambda_1(\Lambda_1-\Lambda_2)},\quad \bar{\gamma}_2=\frac{q_0}{2\omega\Lambda_2(\Lambda_1-\Lambda_2)},$$
$$L^{-1}\frac{q_0\dfrac{p^3}{(p^2+\omega^2)^2}\cdot\dfrac{1}{p^2+2Bp+C^2}}{p}=$$
$$=\int_0^t\{u\,[\sin\omega\,(t-T)+\omega\,(t-T)\cos\omega\,(t-T)]+$$
$$+\,\bar{\gamma}_1 e^{\Lambda_1 T}\,[\sin\omega\,(t-T)+\omega\,(t-T)\cos\omega\,(t-T)]-$$
$$-\,\bar{\gamma}_2 e^{-\Lambda_2 T}\,[\sin\omega\,(t-T)+\omega\,(t-T)\cos\omega\,(t-T)]\}\,dT.$$

As a general conclusion, it can be stated that the vibrations resulting from the effect of the control are attenuated much more rapidly the greater the damping of the natural frequency of the blade.

In the case of variations due to sudden application of control or gradual oblique application in the collective angle of attack, it is found that special conditions of damping of the blades must be satisfied. /1087

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